Naval Surface Warfare Center
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Materials Research and Development Directorate

Technical Report

Boundary Conditions for Magnetic Field Gradients Inside a Source-Free Volume

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Dr. John J. Holmes

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Magnetic potentials, potential gradients, and combinations of the two serve as boundary conditions for solutions to Laplace's equation. From these boundary conditions unique solutions for magnetic potential and field are obtained inside the volume of space that is enclosed by the boundary. In this paper, sufficiency boundary conditions are derived that ensure unique solutions for gradients in magnetic field inside the bounded volume. The derived general boundary conditions are expressed in terms of directional derivatives of magnetic field.

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I INTRODUCTION

There are several properties of Laplace's equation that have important applications. One of these properties states that if the magnetic field normal to the surface bounding a source-free region of space is zero everywhere over that surface, then all magnetic field components within the volume are zero [1]. Another related property states that if both orthogonal components of magnetic field tangent to the bounding surface of a source-free region are zero everywhere over that surface, then all magnetic field components within the volume are once again zero [2]. The purpose of this paper is to derive two analogous properties for special types of magnetic field gradients called "directional derivatives". Once these properties have been established, the boundary conditions ensuring uniqueness for gradient solutions inside the volume can be determined.

There are many ways to compute the gradient of a magnetic scalar potential or field component. In general, the gradient of a magnetic scalar potential, φ, is expressed as $\nabla\varphi$, and is equal to the negative of a magnetic field vector, $\overline{H}$. In Cartesian coordinates the potential gradient is expressed as

$$\nabla\varphi = \hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z} \tag{1}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors in the x, y, z directions, respectively. In generalized curvilinear coordinates (1) must be modified to account for the curvature of the coordinate system. In an orthogonal curvilinear system (OCS), (1) is written as

$$\nabla\varphi = \hat{a}_1\frac{\partial\varphi}{h_1\partial\xi_1} + \hat{a}_2\frac{\partial\varphi}{h_2\partial\xi_2} + \hat{a}_3\frac{\partial\varphi}{h_3\partial\xi_3} \tag{2}$$

where $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$ are unit vectors in the direction of generalized coordinates ξ_1, ξ_2, ξ_3; and where h_1, h_2, h_3 are the metrics that scale the curvilinear coordinate system under consideration [3]. The rate-of-change in each component of the vector in (2) is computed along a curved coordinate line while holding the other coordinates constant.

To take the process one step further, the gradient of each component in $\overline{H}$ could also be computed. In Cartesian coordinates, the complete gradient of $\overline{H}$ is given as

$$\nabla H_x = \hat{i}\frac{\partial H_x}{\partial x} + \hat{j}\frac{\partial H_x}{\partial y} + \hat{k}\frac{\partial H_x}{\partial z} \tag{3}$$

$$\nabla H_y = \hat{i}\frac{\partial H_y}{\partial x} + \hat{j}\frac{\partial H_y}{\partial y} + \hat{k}\frac{\partial H_y}{\partial z} \tag{4}$$

$$\nabla H_z = \hat{i}\frac{\partial H_z}{\partial x} + \hat{j}\frac{\partial H_z}{\partial y} + \hat{k}\frac{\partial H_z}{\partial z} \tag{5}$$

where H_x, H_y, H_z are the magnetic field components in Cartesian coordinates; while in generalized coordinates (reference 3, p122, eq. 7.4) the complete gradient would be

$$\nabla H_1 = \hat{a}_1\frac{\partial H_1}{h_1\partial\xi_1} + \hat{a}_2\frac{\partial H_1}{h_2\partial\xi_2} + \hat{a}_3\frac{\partial H_1}{h_3\partial\xi_3} \tag{6}$$

$$\nabla H_2 = \hat{a}_1 \frac{\partial H_2}{h_1 \partial \xi_1} + \hat{a}_2 \frac{\partial H_2}{h_2 \partial \xi_2} + \hat{a}_3 \frac{\partial H_2}{h_3 \partial \xi_3} \tag{7}$$

$$\nabla H_3 = \hat{a}_1 \frac{\partial H_3}{h_1 \partial \xi_1} + \hat{a}_2 \frac{\partial H_3}{h_2 \partial \xi_2} + \hat{a}_3 \frac{\partial H_3}{h_3 \partial \xi_3} \tag{8}$$

where H_1, H_2, H_3 are the three components of magnetic field in generalized coordinates. (It should be noted that, in general, the metrics h_1, h_2, h_3 are themselves functions of ξ_1, ξ_2, ξ_3.) As was the case for the gradient of the magnetic potential, the gradients of the magnetic field vector components in a generalized coordinate system give the gradient along curved lines defined by the curvilinear coordinate. This curvilinear gradient is not the boundary condition being considered in this paper.

In this paper, magnetic field gradient boundary conditions will refer to the directional derivatives of the magnetic field vectors. The directional derivatives give the rate-of-change in magnetic field along a straight line in a specific direction. The directional derivative of the magnetic field vector $\overline{H}$ in the direction of a unit vector $\hat{a}$ is given in Cartesian coordinates by [4]

$$(\hat{a} \cdot \nabla)\overline{H} = \hat{i}(\hat{a} \cdot \nabla H_x) + \hat{j}(\hat{a} \cdot \nabla H_y) + \hat{k}(\hat{a} \cdot \nabla H_z) \tag{9}$$

while, in generalized curvilinear coordinates the expressions are

$$\begin{aligned}[(\hat{a} \cdot \nabla)\overline{H}]_1 = & \frac{a_1}{h_1}\frac{\partial H_1}{\partial \xi_1} + \frac{a_2}{h_2}\frac{\partial H_1}{\partial \xi_2} + \frac{a_3}{h_3}\frac{\partial H_1}{\partial \xi_3} \\ & + \frac{H_2}{h_1 h_2}[a_1 \frac{\partial h_1}{\partial \xi_2} - a_2 \frac{\partial h_2}{\partial \xi_1}] \\ & + \frac{H_3}{h_1 h_3}[a_1 \frac{\partial h_1}{\partial \xi_3} - a_3 \frac{\partial h_3}{\partial \xi_1}]\end{aligned} \tag{10}$$

$$\begin{aligned}[(\hat{a} \cdot \nabla)\overline{H}]_2 = & \frac{a_1}{h_1}\frac{\partial H_2}{\partial \xi_1} + \frac{a_2}{h_2}\frac{\partial H_2}{\partial \xi_2} + \frac{a_3}{h_3}\frac{\partial H_2}{\partial \xi_3} \\ & + \frac{H_3}{h_2 h_3}[a_2 \frac{\partial h_2}{\partial \xi_3} - a_3 \frac{\partial h_3}{\partial \xi_2}] \\ & + \frac{H_1}{h_1 h_2}[a_2 \frac{\partial h_2}{\partial \xi_1} - a_1 \frac{\partial h_1}{\partial \xi_2}]\end{aligned} \tag{11}$$

$$\begin{aligned}[(\hat{a} \cdot \nabla)\overline{H}]_3 = & \frac{a_1}{h_1}\frac{\partial H_3}{\partial \xi_1} + \frac{a_2}{h_2}\frac{\partial H_3}{\partial \xi_2} + \frac{a_3}{h_3}\frac{\partial H_3}{\partial \xi_3} \\ & + \frac{H_1}{h_1 h_3}[a_3 \frac{\partial h_3}{\partial \xi_1} - a_1 \frac{\partial h_1}{\partial \xi_3}] \\ & + \frac{H_2}{h_2 h_3}[a_3 \frac{\partial h_3}{\partial \xi_2} - a_2 \frac{\partial h_2}{\partial \xi_3}]\end{aligned} \tag{12}$$

where a_1, a_2, a_3 are the components of $\hat{a}$ in curvilinear coordinates. The objective of this paper is to determine the magnetic field boundary conditions for directional derivatives that will guarantee all components of the complete magnetic field gradient are uniquely zero inside the bounded volume.

II Boundary Conditions for Directional Derivatives in the Surface-Normal Direction

The standard derivation of the magnetic field properties of Laplace's equation will serve as the bases for proving the gradient cases. Following the procedure given in [1], applying the divergence theorem to $H_x \nabla H_x$ gives

$$\int_V \nabla \cdot (H_x \nabla H_x) dv = \oint_S (H_x \nabla H_x) \cdot \hat{n} da \tag{13}$$

where da is the differential area on the closed surface, S, that bounds the source-free volume, V, having differential volume dv (see Fig. 1), and $\hat{n}$ is the unit normal vector on the bounding surface that points into the volume. However,

$$\nabla \cdot (H_x \nabla H_x) = H_x \nabla^2 H_x + \nabla H_x \cdot \nabla H_x \tag{14}$$

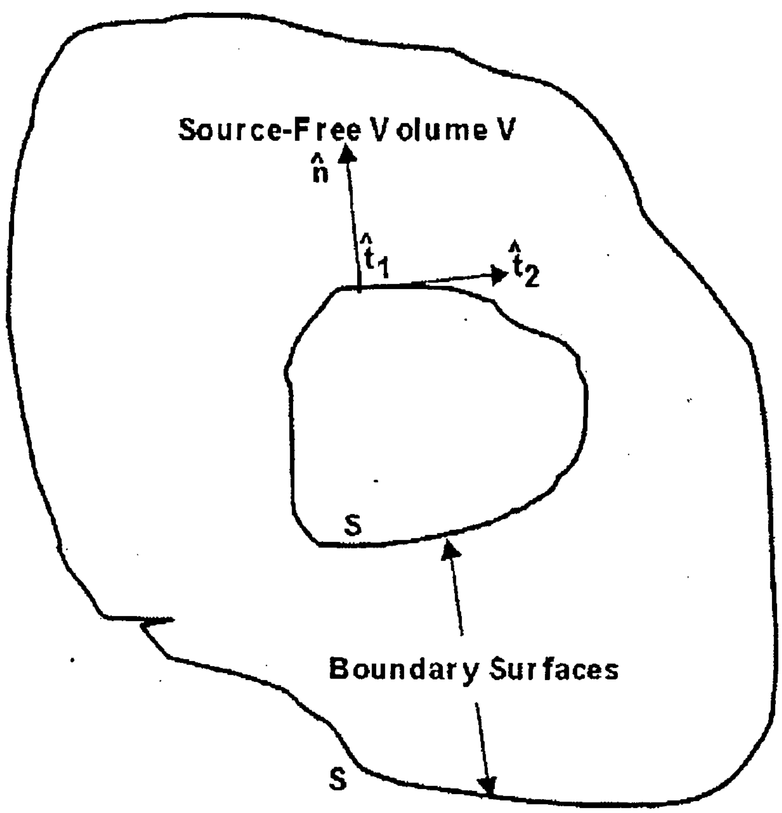

Figure 1. A Source Free Volume, V, bounded by the Surface, S

Since $H_x = -\partial\varphi / \partial x$, then

$$\nabla^2 H_x = -(\frac{\partial^3 \varphi}{\partial^3 x} + \frac{\partial^3 \varphi}{\partial x \partial^2 y} + \frac{\partial^3 \varphi}{\partial x \partial^2 z}) \tag{15}$$

which becomes

$$\frac{\partial}{\partial x}\left(\nabla^2\varphi\right)=0 \tag{16}$$

since φ is a solution to Laplace's equation. (In general, the order of differentiation in (15) can only be reversed in Cartesian coordinates.) Therefore, (14) reduces to $\nabla\cdot\left(H_x\nabla H_x\right)=\nabla H_x\cdot\nabla H_x$, so that (13) can be written as

$$\int_V\left|\nabla H_x\right|^2 dv=\oint_S H_x\left(\hat{n}\cdot\nabla H_x\right)da \tag{17}$$

Repeating this procedure for the y and z components gives

$$\int_V\left|\nabla H_y\right|^2 dv=\oint_S H_y\left(\hat{n}\cdot\nabla H_y\right)da \tag{18}$$

and

$$\int_V\left|\nabla H_z\right|^2 dv=\oint_S H_z\left(\hat{n}\cdot\nabla H_z\right)da \tag{19}$$

Adding (17) through (19) together, and rearranging yields

$$\int_V\left(\left|\nabla H_x\right|^2+\left|\nabla H_y\right|^2+\left|\nabla H_z\right|^2\right)dv =\oint_S \overline{H}\cdot[\left(\hat{n}\cdot\nabla\right)\overline{H}]da \tag{20}$$

The term in brackets on the right side of (20) is the directional derivative of $\overline{H}$ in the direction that is normal to the bounding surface.

If all the components of the directional derivative in the direction normal to the surface are zero, then the right side of (20) is zero. Therefore, the integrand on the left side is also equal to zero. This means that all the magnetic field gradient components (i.e. the complete gradient of $\overline{H}$) are zero inside the volume.

It would appear at first glance that the zero requirements for all components of the directional derivative in the direction normal to the surface applies only to magnetic field vectors expressed in Cartesian coordinates. However, the result is more general. If $\overline{H}$ is expressed as

$$\overline{H}=\hat{n}H_n+\hat{t}_1H_{t_1}+\hat{t}_2H_{t_2} \tag{21}$$

where $\hat{t}_1$ and $\hat{t}_2$ are unit vectors in the two orthogonal directions tangent to the surface (Fig. 1), then from (10) through (12), the three components of the directional derivatives in the normal direction ($D_{nn}, D_{t_1n}, D_{t_2n}$) are given by

$$D_{nn}=\frac{1}{h_n}\frac{\partial H_n}{\partial\xi_n}+\frac{H_{t_1}}{h_nh_{t_1}}\frac{\partial h_n}{\partial\xi_{t_1}}+\frac{H_{t_2}}{h_nh_{t_2}}\frac{\partial h_n}{\partial\xi_{t_2}} \tag{22}$$

$$D_{t_1 n} = \frac{1}{h_n}\frac{\partial H_{t_1}}{\partial \xi_n} - \frac{H_n}{h_n h_{t_1}}\frac{\partial h_n}{\partial \xi_{t_1}} \tag{23}$$

$$D_{t_2 n} = \frac{1}{h_n}\frac{\partial H_{t_2}}{\partial \xi_n} - \frac{H_n}{h_n h_{t_2}}\frac{\partial h_n}{\partial \xi_{t_2}} \tag{24}$$

where h_n, h_{t_1}, h_{t_2} are the metrics for the surface-normal and two orthogonal tangential directions, respectively. Therefore, forcing $D_{nn} = D_{t_1 n} = D_{t_2 n} = 0$ ensures $(\hat{n}\cdot\nabla)\overline{H} = 0$ in (20).

When examining (20), it becomes clear that the requirement for all three components of the directional derivatives to be zero may, in some cases, be too rigid. Expanding the integrand on the left side of (20) gives

$$\begin{aligned} |\nabla H_x|^2 + |\nabla H_y|^2 + |\nabla H_z|^2 &= H_{xx}^2 + H_{xy}^2 + H_{xz}^2 \\ &\quad + H_{yx}^2 + H_{yy}^2 + H_{yz}^2 \\ &\quad + H_{zx}^2 + H_{zy}^2 + H_{zz}^2 \end{aligned} \tag{25}$$

However, since $H_{zx} = H_{xz}$, $H_{zy} = H_{yz}$, and $H_{zz} = -(H_{xx} + H_{yy})$, it needs only be shown that $|\nabla H_x|^2 + |\nabla H_y|^2 = 0$ to prove that the complete gradient inside the volume is zero. In fact, setting any two of the three terms on the left side of (25) to zero is sufficient for the complete gradient to be zero; which implies that in Cartesian coordinates only two of the three gradient boundary conditions are required. There are other coordinate systems in which only two of the three components of the normal directional derivatives are required. Therefore, the general boundary condition specifying all three components of the directional derivatives in the surface-normal direction is a sufficient condition, but not a necessary one.

Equation (20) can now be used to prove the uniqueness of solutions to Laplace's equations for field gradient boundary conditions. Assume there are two possible solutions to Laplace's equations for field gradients inside the volume V, $\overline{H}_a$ and $\overline{H}_b$. Let $\overline{H}^d = \overline{H}_a - \overline{H}_b$, so that $(\hat{n}\cdot\nabla)\overline{H}_a - (\hat{n}\cdot\nabla)\overline{H}_b = 0$ on the bounding surface. (It is a requirement that both solutions satisfy the same boundary conditions on the enclosing surface.) Therefore, since $(\hat{n}\cdot\nabla)\overline{H}^d = 0$, $|\nabla H_x^d|^2 + |\nabla H_y^d|^2 + |\nabla H_z^d|^2 = 0$ inside the volume. This implies that the complete gradients of $\overline{H}_a$ and $\overline{H}_b$ are equal in this region, proving uniqueness of the solution.

III Boundary Conditions for Directional Derivatives in the Surface-Tangential Directions

The standard derivation of the tangential magnetic field boundary conditions for Laplace's equation will serve as the bases for proving the gradient cases. Following the procedure given in [2], the vector analogue of Green's first identity is

$$\begin{aligned} &\int_V \left(\nabla\times\overline{P}\cdot\nabla\times\overline{Q} - \overline{P}\cdot\nabla\times\nabla\times\overline{Q}\right)dv \\ &\qquad = \oint_S \left(\overline{P}\times\nabla\times\overline{Q}\right)\cdot\hat{n}\,da \end{aligned} \tag{26}$$

where as before S is a closed surface around the volume V. Let $\nabla H_x = \nabla \times \overline{P}^x = \nabla \times \overline{Q}^x$ (which implies $\nabla \times \nabla \times \overline{Q}^x = 0$), where $\overline{P}^x$ and $\overline{Q}^x$ are given by $\overline{P}^x = \hat{a}_1 P_1^x + \hat{a} P_2^x + \hat{a}_3 P_3^x$ and $\overline{Q}^x = \hat{a}_1 Q_1^x + \hat{a}_2 Q_2^x + \hat{a}_3 Q_3^x$. (The vector $\overline{P}^x$ is shown to exist and is defined in the appendix for Cartesian coordinates.) Rearranging the right side of (26) gives

$$\int_V |\nabla H_x|^2 dv = \oint_S \overline{P}^x \cdot (\nabla H_x \times \hat{n}) da \tag{27}$$

Substituting $\hat{n} = \hat{t}_1 \times \hat{t}_2$ in (27) and using a vector identity from [4] gives

$$\int_V |\nabla H_x|^2 dv = \oint_S \overline{P}^x \cdot \left[(\hat{t}_2 \cdot \nabla H_x) \hat{t}_1 - (\hat{t}_1 \cdot \nabla H_x) \hat{t}_2 \right] da \tag{28}$$

Repeating this procedure for the y and z components gives

$$\int_V |\nabla H_y|^2 dv = \oint_S \overline{P}^y \cdot \left[(\hat{t}_2 \cdot \nabla H_y) \hat{t}_1 - (\hat{t}_1 \cdot \nabla H_y) \hat{t}_2 \right] da \tag{29}$$

and

$$\int_V |\nabla H_z|^2 dv = \oint_S \overline{P}^z \cdot \left[(\hat{t}_2 \cdot \nabla H_z) \hat{t}_1 - (\hat{t}_1 \cdot \nabla H_z) \hat{t}_2 \right] da \tag{30}$$

The terms inside the brackets on the right side of (28) through (30) are the directional derivatives of H_x, H_y, H_z in the two orthogonal directions tangent to the bounding surface. If these six directional derivatives are all zero, then the complete gradient within the volume is also zero.

Once again it can be shown that the directional derivative boundary conditions apply to curvilinear coordinate systems. The six tangential directional derivative boundary conditions from (28) through (30) can be written as

$$(\hat{t}_1 \cdot \nabla) \overline{H} = \hat{i}(\hat{t}_1 \cdot \nabla H_x) + \hat{j}(\hat{t}_1 \cdot \nabla H_y) + \hat{k}(\hat{t}_1 \cdot \nabla H_z) \tag{31}$$

$$(\hat{t}_2 \cdot \nabla) \overline{H} = \hat{i}(\hat{t}_2 \cdot \nabla H_x) + \hat{j}(\hat{t}_2 \cdot \nabla H_y) + \hat{k}(\hat{t}_2 \cdot \nabla H_z) \tag{32}$$

Once again, if $\overline{H}$ is expressed as in (21), then from (10) through (12), the six components of the directional derivatives in the tangential direction ($D_{nt_1}, D_{t_1t_1}, D_{t_2t_1}, D_{nt_2}, D_{t_1t_2}, D_{t_2t_2}$) are given by

$$D_{nt_1} = \frac{1}{h_{t_1}} \frac{\partial H_n}{\partial \xi_{t_1}} - \frac{H_{t_1}}{h_n h_{t_1}} \frac{\partial h_{t_1}}{\partial \xi_n} \tag{33}$$

$$D_{t_1t_1} = \frac{1}{h_{t_1}} \frac{\partial H_{t_1}}{\partial \xi_{t_1}} + \frac{H_{t_2}}{h_{t_1} h_{t_2}} \frac{\partial h_{t_1}}{\partial \xi_{t_2}} + \frac{H_n}{h_n h_{t_1}} \frac{\partial h_{t_1}}{\partial \xi_n} \tag{34}$$

$$D_{t_2t_1} = \frac{1}{h_{t_1}} \frac{\partial H_{t_2}}{\partial \xi_{t_1}} - \frac{H_{t_1}}{h_{t_1}h_{t_2}} \frac{\partial h_{t_1}}{\partial \xi_{t_2}} \tag{35}$$

$$D_{nt_2} = \frac{1}{h_{t_2}} \frac{\partial H_n}{\partial \xi_{t_2}} - \frac{H_{t_2}}{h_n h_{t_2}} \frac{\partial h_{t_2}}{\partial \xi_n} \tag{36}$$

$$D_{t_1t_2} = \frac{1}{h_{t_2}} \frac{\partial H_{t_1}}{\partial \xi_{t_2}} - \frac{H_{t_2}}{h_{t_1}h_{t_2}} \frac{\partial h_{t_2}}{\partial \xi_{t_1}} \tag{37}$$

$$D_{t_2t_2} = \frac{1}{h_{t_2}} \frac{\partial H_{t_2}}{\partial \xi_{t_2}} + \frac{H_n}{h_n h_{t_2}} \frac{\partial h_{t_2}}{\partial \xi_n} + \frac{H_{t_1}}{h_{t_1}h_{t_2}} \frac{\partial h_{t_2}}{\partial \xi_{t_1}} \tag{38}$$

where all the terms have been defined previously. Therefore, if $D_{nt_1} = D_{t_1t_1} = D_{t_2t_1} = D_{nt_2} = D_{t_2t_1} = D_{nt_2} = 0$, then $(\hat{t}_1 \cdot \nabla)\overline{H} = (\hat{t}_2 \cdot \nabla)\overline{H} = 0$.

Using the same arguments as for the surface-normal directional derivatives, it can be shown that all six components are not required for a Cartesian system, as is the case in some curvilinear coordinate systems. Therefore, the requirement that all six tangential directional derivatives be zero is a sufficient condition and not a necessary one. In addition, these six directional derivatives are sufficient to guarantee a unique complete gradient solution to Laplace's equation.

IV CONCLUSION

Specifying the three magnetic field directional derivatives in the normal direction over a closed surface around a source-free volume is a sufficient, but not necessary, condition to ensure a unique total field-gradient solution to Laplace's equation. Similarly, specifying the six magnetic field directional derivatives in the two orthogonal tangential directions over a closed surface around a source-free volume is also a sufficient, but not necessary, condition to ensure a unique total field-gradient solution to Laplace's equation. Although the proofs in this paper were conducted specifically for magnetic fields, the directional derivative boundary conditions should apply to any field-gradient boundary value problem in physics. In addition, it may be possible to use this theory to develop finite element techniques that use directional derivatives as input to boundary value problems and compute unique solutions for the complete gradient inside a volume.

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APPENDIX

In this appendix it will be shown that a vector $\overline{P}^x$ can be found such that $\nabla H_x = \nabla \times \overline{P}^x$. The magnetic field $\overline{H}$ can be represented as the curl of a vector potential $\overline{A}$ ($\overline{H} = \nabla \times \overline{A}$) and can be written in Cartesian coordinates as

$$\overline{H} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \tag{A1}$$

where A_x, A_y, A_z are the components of $\overline{A}$. From (A1), the expression for ∇H_x is

$$\begin{aligned}\nabla H_x = \hat{i}(\frac{\partial^2 A_z}{\partial y \partial x} - \frac{\partial^2 A_y}{\partial z \partial x}) + \hat{j}(\frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_y}{\partial z \partial y}) \\ + \hat{k}(\frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial z^2})\end{aligned} \tag{A2}$$

Next, $\nabla \times \overline{P}^x$ is given by

$$\begin{aligned}\nabla \times \overline{P}^x = \hat{i}(\frac{\partial P_z^x}{\partial y} - \frac{\partial P_y^x}{\partial z}) + \hat{j}(\frac{\partial P_x^x}{\partial z} - \frac{\partial P_z^x}{\partial x}) \\ + \hat{k}(\frac{\partial P_y^x}{\partial x} - \frac{\partial P_x^x}{\partial y})\end{aligned} \tag{A3}$$

Equating like components between (A2) and (A3) gives the three equations

$$\frac{\partial P_z^x}{\partial y} - \frac{\partial P_y^x}{\partial z} = \frac{\partial^2 A_z}{\partial y \partial x} - \frac{\partial^2 A_y}{\partial z \partial x} \tag{A4}$$

$$\frac{\partial P_x^x}{\partial z} - \frac{\partial P_z^x}{\partial x} = \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_y}{\partial z \partial y} \tag{A5}$$

$$\frac{\partial P_y^x}{\partial x} - \frac{\partial P_x^x}{\partial y} = \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial z^2} \tag{A6}$$

From (A4) it is clear that P_y^x and P_z^x can be represented as

$$P_y^x = \frac{\partial A_y}{\partial x} \tag{A7}$$

$$P_z^x = \frac{\partial A_z}{\partial x} \tag{A8}$$

since the order of differentiation can be changed in Cartesian coordinates. What remains is to show that the selection of P_y^x and P_z^x in (A7) and (A8) are consistent with (A5) and (A6). Placing (A8) into (A5) and taking the partial derivative with respect to y gives

$$\frac{\partial^2 P_x^x}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2}\right) - \frac{\partial^3 A_y}{\partial y^2 \partial z} \tag{A9}$$

However, H_x is also a solution to Laplace's equation as shown by (15) and (16) of the main text, and can be written from (A1) as

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \\ + \frac{\partial^2}{\partial z^2}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) = 0\end{aligned} \tag{A10}$$

which can be rearranged to give

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right) \\ = \frac{\partial}{\partial z}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\end{aligned} \tag{A11}$$

Using (A11) with (A9) and reducing produces

$$\frac{\partial^2 P_x^x}{\partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_z}{\partial y \partial z}\right) \tag{A12}$$

Placing (A7) into (A6) and taking the partial derivative with respect to z gives an expression that is identical to (A12). Therefore, the choice of P_y^x and P_z^x given in (A7) and (A8) are consistent with (A4) through (A6).

In this appendix, it has been shown that $\overline{P}^x$ exists such that $\nabla H_x = \nabla \times \overline{P}^x$. In addition, the expressions for the three Cartesian components of $\overline{P}^x$ have been determined. Similarly, it can be shown that $\overline{P}^y$ and $\overline{P}^z$ exist such that $\nabla H_y = \nabla \times \overline{P}^y$ and $\nabla H_z = \nabla \times \overline{P}^z$.

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